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Quantum energy inequalities for the non-minimally coupled scalar field

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Abstract
In this paper, we discuss local averages of the energy density for the non-minimally coupled scalar quantum field, extending a previous investigation of the classical field. By an explicit example, we show that such averages are unbounded from below on the class of Hadamard states. This contrasts with the minimally coupled field, which obeys a state-independent lower bound known as a quantum energy inequality (QEI). Nonetheless, we derive a generalized QEI for the non-minimally coupled scalar field, in which the lower bound is permitted to be state-dependent. This result applies to general globally hyperbolic curved spacetimes for coupling constants in the range $0 < \xi \leqslant 1/4$. We analyse the state dependence of our QEI in four-dimensional Minkowski space and show that it is a non-trivial restriction on the averaged energy density in the sense that the lower bound is of lower order, in energetic terms, than the averaged energy density itself.

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1. Introduction

For more than 30 years it has been known that the stress–energy tensor of the classical scalar field, obtained from the Lagrangian

$$L[\phi, g] = \tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{2}(m^2 + \xi R)\phi^2 \tag{1}$$

does not satisfy the weak energy condition (WEC) at non-minimal coupling, i.e., $\xi \neq 0$ (see [1, 2] for a simple example in Minkowski space). Naturally, this raises the question of whether there are any restrictions on the extent of WEC violation and whether this field could support exotic phenomena (e.g., violations of the second law of thermodynamics) that depend on macroscopic spacetime regions of negative energy density.

We showed in [3] that under certain conditions one can find lower bounds for local averages of the energy density of the classical non-minimally coupled scalar field. This paper extends our analysis to the case of the quantized field.

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As is well known, quantum field theories obeying the Wightman axioms necessarily violate the WEC [4] and in this respect, the non-minimally coupled scalar field resembles the situation at minimal coupling. However, as we show, there are differences. For example, consider the case where $\xi > 0$. Given any bounded subset $\mathcal{O}$ of Minkowski space and an arbitrary constant $\rho_0 > 0$ we will construct a Hadamard state Ψ in which the expected energy density $\langle\rho\rangle_\Psi$ is less than $-\rho_0$ throughout $\mathcal{O}$. Therefore, non-trivial local averages of the energy density, $\langle\rho(\mathfrak{f})\rangle_\Psi$, where $\mathfrak{f}$ is a non-negative test function, are unbounded from below on the class of Hadamard states.

By contrast, expectation values of the averaged energy density of the *minimally* coupled scalar field are bounded from below on the class of Hadamard states [5]. The latter bound, known as a quantum energy inequality (QEI), can be written in the form

$$\langle\rho(\mathfrak{f})\rangle_\Psi \geqslant -\widetilde{\mathfrak{Q}}(\mathfrak{f}) \tag{2}$$

for all Hadamard states Ψ, where $\widetilde{\mathfrak{Q}}(\mathfrak{f})$ is a constant (see [6–8] for reviews and further references concerning such QEIs). Clearly, the non-minimally coupled field cannot satisfy a QEI of this type. However, we will show that it obeys a generalized QEI of the form

$$\langle\rho(\mathfrak{f})\rangle_\Psi \geqslant -\langle\mathfrak{Q}(\mathfrak{f})\rangle_\Psi \tag{3}$$

for all Hadamard states Ψ, where $\mathfrak{Q}(\mathfrak{f})$ is now allowed to be an unbounded operator, which turns out to involve the Wick square of the field. This bound will be proved for averaging along time-like geodesics in general globally hyperbolic spacetimes. Precise statements and the proof are given in section 4. State-dependent QEIs (and related results) have recently been studied in an abstract context by one of us [9], in which they are naturally suggested by the mathematical framework. This paper complements that work by giving a concrete example of a quantum field which admits a state-dependent bound but cannot admit a state-independent one.

The state-dependent nature of the lower bound raises an important question. It is clear that setting $\mathfrak{Q}(\mathfrak{f}) = -\rho(\mathfrak{f})$ would provide a rather trivial inequality of the above type. Are our bounds similarly trivial? In section 5, we will analyse this question in two ways. The first is based on a proposal in [9], in which a QEI would be declared trivial if there exist constants c and c' such that

$$|\langle\rho(\mathfrak{f})\rangle_\Psi| \leqslant c + c'|\langle\mathfrak{Q}(\mathfrak{f})\rangle_\Psi| \tag{4}$$

for all Hadamard Ψ. We will show that our bound is non-trivial in this sense by considering finite-temperature states in Minkowski space. The second way uses so-called *H*-bounds: we show that $\mathfrak{Q}(\mathfrak{f})$ can be bounded by any power of the Hamiltonian greater than 2, while $\rho(\mathfrak{f})$ cannot be bounded by powers less than 3.[1] Thus, although the lower bound is state-dependent, it is more stringent in energetic terms than any upper bound on the averaged energy density: one may say that the $\mathfrak{Q}(\mathfrak{f})$ is of lower order than $\rho(\mathfrak{f})$. In particular, states that exhibit large negative energy densities over extended spacetime regions necessarily have large positive overall energy. Further comments on the significance of our results are given in section 6.

2. The non-minimally coupled field

We begin by recalling the definition of the non-minimally coupled scalar field, its quantization and the construction of the stress–energy tensor. This will serve to fix our conventions.

[1] The power of 3 emerges by considering a particular family of states, and it may be that $\rho(\mathfrak{f})$ can only be bounded by powers of at least 4, as would be natural on dimensional grounds. See section 5.2 for more discussion.

The classical Lagrangian describing the field on a n-dimensional spacetime[2] $\mathbf{M} = (M, g)$ is given by (1), where m, ξ are real constants and R is the Ricci scalar with respect to the metric g. The constant ξ is called the coupling constant. If $\xi = 0$, the field is said to be minimally coupled. For $\xi = (n-2)/(4n-4)$ and $\xi = 1/4$, one speaks of conformal and supersymmetric coupling, respectively. In this paper, we will focus on values $\xi \in [0, 1/4]$, which clearly contains all the special values just mentioned. The Lagrangian (1) leads to the wave equation

$$P_\xi \phi = 0, \tag{5}$$

where $P_\xi := \Box_g + (m^2 + \xi R)$ is the Klein–Gordon operator and $\Box_g$ is the d'Alembertian with respect to the metric g. We will follow the standard convention and denote the space of compactly supported, smooth, complex-valued functions on M by $\mathscr{D}(M)$. Assuming that the spacetime is globally hyperbolic, there is an antisymmetric bi-distribution $E_\xi(x, y)$ which is the difference of the advanced and retarded Green functions.

The theory is quantized by introducing a unital *-algebra $\mathfrak{A}_\xi(\mathbf{M})$, which is generated by objects $\Phi(f)$ $(f \in \mathscr{D}(M))$ subject to the relations that (a) the map $f \to \Phi(f)$ is complex linear; (b) $\Phi(f)^* = \Phi(\overline{f})$; (c) $\Phi(P_\xi f) = 0$; (d) $[\Phi(f), \Phi(h)] = \mathrm{i}E_\xi(f, h)\mathbb{1}$ for all $f, h \in \mathscr{D}(M)$. Properties (c) and (d) enforce the field equation and the canonical commutation relations, respectively. In this framework, states are positive and normalized linear functionals $\langle\cdot\rangle_\Psi \to \mathbb{C}$ on the algebra $\mathfrak{A}_\xi(\mathbf{M})$. In particular, we will be interested in Hadamard states: in such a state Ψ, the two-point function $\omega_2^\Psi(x, y) = \langle\Phi(x)\Phi(y)\rangle_\Psi$ is a distribution with a prescribed singularity structure so that the difference between the two-point functions of any two Hadamard states is smooth. See [12] and references therein, for details on Hadamard states. Some of our later results will be based on a characterization of the Hadamard states in terms of microlocal analysis due to Radzikowski [13].

We now turn to the problem of quantizing quadratic classical expressions of the form $G^{\text{class}}(x) = \left[\sum_i \hat{D}_i(\phi\otimes\phi)\right]_c(x)$, where the $\hat{D}_i$ are linear differential operators on $C^\infty(M\times M)$ with smooth coefficients and $[\cdot]_c(x)$ denotes the 'coincidence limit' $[F]_c(x) := F(x, x)$, of any smooth function $F \in C^\infty(M \times M)$. The quantized normal ordered form of G^{class} in the Hadamard state Ψ is then defined by

$$\langle G^{\text{quant}}\rangle_\Psi(x) = \left[\sum_i \hat{D}_i{:}\omega_2^\Psi{:}\right]_c(x), \tag{6}$$

where $:\omega_2^\Psi: = \omega_2^\Psi - \omega_2^0$ is the normal ordering of ω_2^Ψ with respect to a reference Hadamard state ω_0. In Minkowski space one has the distinguished vacuum state Ω and therefore one usually chooses $\omega_2^0 = \omega_2^\Omega$. In all quasi-free representations, normal ordering coincides with Wick normal ordering of annihilation and creation operators.

The classical stress–energy tensor of the non-minimally coupled scalar field can be calculated by varying the action of the Lagrangian (1) with respect to the metric, and takes the form

$$T_{\mu\nu}^{\text{class}} = (\nabla_\mu\phi)(\nabla_\nu\phi) + \tfrac{1}{2}g_{\mu\nu}(m^2\phi^2 - (\nabla\phi)^2) + \xi(g_{\mu\nu}\Box_g - \nabla_\mu\nabla_\nu - G_{\mu\nu})\phi^2, \tag{7}$$

where $G_{\mu\nu}$ is the Einstein tensor. The term proportional to the coupling constant ξ originates in the variation of the coupling term in the Lagrangian density. In a Ricci-flat spacetime, the differential operators in (7) proportional to ξ are still present. So minimal coupling and vanishing Ricci scalar result in the same wave equation but in a different stress–energy tensor.

[2] Our sign conventions are those of Birrell and Davies [10], i.e., the $[-, -, -]$ convention in the classification scheme of Misner, Thorne and Wheeler [11].

To quantize (7) in the way we introduced above, we need to bring it into the form used in (6). This can be done with the definition of the Klein–Gordon operator P_ξ. One finds that

$$T^{\text{class}}_{\mu\nu} = (1-2\xi)(\nabla_\mu\phi)(\nabla_\nu\phi) + \tfrac{1}{2}(1-4\xi)g_{\mu\nu}(m^2\phi^2 - (\nabla\phi)^2) \\ - 2\xi\left(\phi\nabla_\mu\nabla_\nu\phi + \tfrac{1}{2}R_{\mu\nu}\phi^2 - \tfrac{1}{4}(1-4\xi)g_{\mu\nu}R\phi^2 - g_{\mu\nu}\phi P_\xi\phi\right). \quad (8)$$

The last term in the bottom line vanishes 'on shell', that is for ϕ satisfying the wave equation (5).

We shall often study the energy density of (8) with respect to freely falling observers. Assume that γ is a time-like geodesic parameterized by proper time[3], i.e., $\dot\gamma^2 = 1$ and $\nabla_{\dot\gamma}\dot\gamma = \dot\gamma^\mu\nabla_\mu\dot\gamma = 0$. Using this, together with (8), one can show that the classical energy density $\rho^{\text{class}}_\phi = T^{\text{class}}_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu$ on γ is

$$\rho^{\text{class}}_\phi = (1-2\xi)(\nabla_{\dot\gamma}\phi)^2 + \tfrac{1}{2}(1-4\xi)(m^2\phi^2 - (\nabla\phi)^2) \\ - 2\xi\left(\phi\nabla^2_{\dot\gamma}\phi + \tfrac{1}{2}R_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu\phi^2 - \tfrac{1}{4}(1-4\xi)R\phi^2 - \phi P_\xi\phi\right). \quad (9)$$

Now let $\mathcal{T}$ be an open tubular neighbourhood of γ. Take a family of smooth vector fields $\{v_i\}_{i=0,\dots,n-1}$ on $\mathcal{T}$, whose restriction to γ is a vielbein with the property that $v_0|_\gamma = \dot\gamma$, so we have $g^{\mu\nu}|_\gamma = v_0^\mu v_0^\nu - \sum_{i=1}^{n-1} v_i^\mu v_i^\nu$. We now introduce the operators

$$\hat\rho_1 = \frac{1}{2}\left(\nabla_{v_0}\otimes\nabla_{v_0}\right) + \frac{1}{2}(1-4\xi)\left(m^2(\mathbb{1}\otimes\mathbb{1}) + \sum_{i=1}^{n-1}\left(\nabla_{v_i}\otimes\nabla_{v_i}\right)\right), \quad (10)$$

$$\hat\rho_2 = 2\left(\mathbb{1}\otimes_{\mathfrak{s}}\nabla^2_{v_0}\right), \quad (11)$$

$$\hat\rho_3 = -\left(\mathbb{1}\otimes_{\mathfrak{s}} R_{\mu\nu}v_0^\mu v_0^\nu\right)\mathbb{1}\Big) + \frac{1}{2}(1-4\xi)\left(\mathbb{1}\otimes_{\mathfrak{s}} R\mathbb{1}\right) + 2(\mathbb{1}\otimes_{\mathfrak{s}} P_\xi). \quad (12)$$

Here, $\otimes_{\mathfrak{s}}$ is the symmetrized tensor product, i.e., $P\otimes_{\mathfrak{s}} P' = \{(P\otimes P') + (P'\otimes P)\}/2$. Having introduced these operators, one finds that

$$\rho^{\text{class}}_\phi = [\hat\rho(\phi\otimes\phi)]_c\,, \quad (13)$$

$$\hat\rho = \hat\rho_1 - \xi\hat\rho_2 + \xi\hat\rho_3. \quad (14)$$

Note that $\hat\rho_3(\phi\otimes\phi) = 0$ in Ricci-flat spacetimes if ϕ is a solution to the wave equation (5). Furthermore, for minimal coupling ($\xi = 0$), we have $\hat\rho = \hat\rho_1$.

We quantize the energy density by replacing the classical point-split field $\phi\otimes\phi$ by the normal ordered two-point function $:\omega_2^\Psi:$ of some Hadamard state Ψ. As noted before, normal ordering is always performed with respect to some fixed reference Hadamard state Ψ_0. So the quantized energy density on γ in the state Ψ is simply given by $\langle\rho^{\text{quant}}\rangle_\Psi = \left[\hat\rho{:}\omega_2^\Psi{:}\right]_c$. Note that our normal ordered energy density is not the same as the renormalized energy density $\langle\rho^{\text{ren}}\rangle_\Psi$ obtained using the Hadamard prescription (see, e.g., [14]) but they are related by $\langle\rho^{\text{quant}}\rangle_\Psi = \langle\rho^{\text{ren}}\rangle_\Psi - \langle\rho^{\text{ren}}\rangle_{\Psi_0}$.

We end this section with a short summary of the non-minimally coupled scalar quantum field in the n-dimensional Minkowski space $\mathbf{M}^n_{\text{Mink}} = (\mathbb{R}^n, \eta)$. We define the measure $\mathrm{d}\mu(\mathbf{k})$ by

$$\mathrm{d}\mu(\mathbf{k}) = \frac{\mathrm{d}^{n-1}\mathbf{k}}{(2\pi)^{n-1}}\frac{1}{2\omega(\mathbf{k})}, \quad (15)$$

[3] We require γ to be connected, but it does not have to be inextendible.

with $\omega(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$ and use it to define the one-particle Hilbert space by $\mathcal{H} = L^2(\mathbb{R}^{n-1}, \mathrm{d}\mu(\mathbf{k}))$. We will denote the norm and inner product on $\mathcal{H}$ by $\|\cdot\|_{\mathcal{H}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, and define the bosonic Fock-space $\mathcal{F}_s(\mathcal{H})$ in the usual way. Thus, for each $g \in \mathcal{H}$, we have an annihilation operator $a(g)$ and creation operator $a^\dagger(g)$ with common domain $D((N+\mathbb{1})^{1/2})$, where N is the number operator on $\mathcal{F}_s(\mathcal{H})$, obeying the canonical commutation relations $[a(f), a^\dagger(g)] = \langle f, g\rangle_{\mathcal{H}}\mathbb{1}$ (recall that $g \mapsto a(g)$ is antilinear).

For any compactly supported distribution $f \in \mathscr{E}'(\mathbf{M}^n_{\mathrm{Mink}})$, we define $\widetilde{f}(\mathbf{k}) = \hat{f}(\omega(\mathbf{k}), \mathbf{k})$, with the Fourier transformation convention

$$\hat{f}(k) = \int \mathrm{d}^n x\, \mathrm{e}^{\mathrm{i}kx} f(x). \tag{16}$$

If f additionally satisfies the property that $\|\widetilde{f}\|_{\mathcal{H}} < \infty$ and $\|\widetilde{\overline{f}}\|_{\mathcal{H}} < \infty$, we can define

$$\Phi(f) = a(\widetilde{\overline{f}}) + a^\dagger(\widetilde{f}) \tag{17}$$

as an operator on $D((N+\mathbb{1})^{1/2})$. If we restrict to smooth compactly supported f, the operators $\Phi(f)$ restricted to $\cap_{k=0}^{\infty} D(N^k)$ generate a representation of $\mathfrak{A}_\xi(\mathbf{M}^n_{\mathrm{Mink}})$ (for any ξ).

Formally, we may write

$$a(g) = \int \mathrm{d}\mu(\mathbf{k})\overline{g}(\mathbf{k})a(\mathbf{k}) \qquad \text{and} \qquad a^\dagger(g) = \int \mathrm{d}\mu(\mathbf{k})g(\mathbf{k})a^\dagger(\mathbf{k}) \tag{18}$$

with the $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ satisfying the commutation relations

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^{n-1} 2\omega(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')\mathbb{1}, \tag{19}$$

and the smeared field may be written as

$$\Phi(f) = \int \mathrm{d}^n x\, \Phi(x) f(x), \tag{20}$$

where

$$\Phi(x) = \int \mathrm{d}\mu(\mathbf{k})(a(\mathbf{k})\,\mathrm{e}^{-\mathrm{i}kx} + a^\dagger(\mathbf{k})\,\mathrm{e}^{\mathrm{i}kx}), \tag{21}$$

with $x = (t, \mathbf{x})$ and $k = (\omega(\mathbf{k}), \mathbf{k})$. Expressions of this type can be made rigorous (cf section X.7 in [15]) and we will make use of this later on.

3. Quantum states with negative energy density

In this section, we consider the massless quantized scalar quantum field with $\xi > 0$ in a (3 + 1)-dimensional Minkowski space $\mathbf{M}^n_{\mathrm{Mink}}$. It will be shown that local averages of the energy density are unbounded from below on the class of Hadamard states.

We start by considering one-particle states Ψ, for which we have the identity

$$:\omega_2^\Psi:(t, \mathbf{x}, t', \mathbf{x}') = \langle\Psi|{:}\Phi(t,\mathbf{x})\Phi(t',\mathbf{x}'){:}\Psi\rangle = 2\mathrm{Re}(\overline{\langle\Omega|\Phi(t,\mathbf{x})\Psi\rangle}\langle\Omega|\Phi(t',\mathbf{x}')\Psi\rangle), \tag{22}$$

as can be shown by writing the fields in terms of annihilation and creation operators. The expression for the renormalized energy density becomes relatively simple since the differential operator $\hat{\rho}$, defined in (14), acting on bi-solutions in Minkowski space is of the form

$$\hat{\rho} = \frac{1}{2}(\partial_t \otimes \partial_{t'}) + \frac{1}{2}(1-4\xi)\sum_{b=1}^{3}(\partial_b \otimes \partial_{b'}) - \xi\left(\left(\partial_t^2 \otimes \mathbb{1}\right) + \left(\mathbb{1} \otimes \partial_{t'}^2\right)\right). \tag{23}$$

So, using (22) we have

$$\langle\Psi|\rho^{\text{quant}}\Psi\rangle(t,\mathbf{x}) = |\partial_t\langle\Omega|\Phi(t,\mathbf{x})\Psi\rangle|^2 + (1-4\xi)\sum_{b=1}^{3}|\partial_b\langle\Omega|\Phi(t,\mathbf{x})\Psi\rangle|^2 - 4\xi\,\text{Re}\left(\overline{\langle\Omega|\Phi(t,\mathbf{x})\Psi\rangle}\times\partial_t^2\langle\Omega|\Phi(t,\mathbf{x})|\Psi\rangle\right). \tag{24}$$

For $\kappa > 0$, we consider (normalized) one-particle states of the form $\Psi_\kappa = a^\dagger(h_\kappa)\Omega$, where $h_\kappa(\mathbf{k}) = 4\pi\sqrt{2}(\kappa - |\mathbf{k}|/3)\,\text{e}^{-|\mathbf{k}|/\kappa}/\kappa^2$, for which

$$\langle\Omega|\Phi(t,\mathbf{x})\Psi\rangle = \int \text{d}\mu(\mathbf{k})\,\text{e}^{-\text{i}(t|\mathbf{k}|-\mathbf{xk})}h_\kappa(\mathbf{k}). \tag{25}$$

Owing to the rapid decay of h_κ, Ψ_κ is Hadamard. It has expected energy

$$\langle\Psi_\kappa|H\Psi_\kappa\rangle = \frac{2\kappa}{3}. \tag{26}$$

The functions h_κ obey $h_{\lambda\kappa}(\mathbf{k}) = h_\kappa(\mathbf{k}/\lambda)/\lambda$, as a consequence of which we have the scaling relation

$$\langle\Psi_{\lambda\kappa}|\rho^{\text{quant}}\Psi_{\lambda\kappa}\rangle(x) = \lambda^4\langle\Psi_\kappa|\rho^{\text{quant}}\Psi_\kappa\rangle(\lambda x). \tag{27}$$

Evaluating the energy density of the state $|\Psi_\kappa\rangle$ at the spatial origin (for example), one finds that[4]

$$\langle\Psi_\kappa|\rho^{\text{quant}}\Psi_\kappa\rangle(t,\mathbf{0}) = \frac{8\kappa^4}{3(1+t^2\kappa^2)^5\pi^2}\{(3t^4\kappa^4 + 3t^2\kappa^2) - \xi(18t^4\kappa^4 - 44t^2\kappa^2 + 2)\}, \tag{28}$$

so, in particular,

$$\langle\Psi_\kappa|\rho^{\text{quant}}\Psi_\kappa\rangle(0,\mathbf{0}) = -\xi\frac{(2\kappa)^4}{3\pi^2}. \tag{29}$$

Owing to the continuity of the expected energy density and (29), it follows that to every $\kappa > 0$, there exists a constant $\tau > 0$, such that

$$\langle\Psi_\kappa|\rho^{\text{quant}}\Psi_\kappa\rangle(x) \leqslant -\xi\frac{(2\kappa)^4}{6\pi^2} \qquad \text{for all} \quad x \in B(\tau), \tag{30}$$

where $B(\tau)$ is the open ball

$$B(\tau) = \{(t,\mathbf{x})|t^2 + |\mathbf{x}|^2 < \tau^2\}. \tag{31}$$

Now fix some κ and some appropriate τ. As a consequence of (27) we can find a $\kappa' > 0$ to every (arbitrary) $\tau' > 0$, such that

$$\langle\Psi_{\kappa'}|\rho^{\text{quant}}\Psi_{\kappa'}\rangle(x) \leqslant -\xi\frac{(2\kappa')^4}{6\pi^2} \qquad \text{for all} \quad x \in B(\tau'). \tag{32}$$

In particular, we can take $\kappa' = \kappa\tau/\tau'$. We have therefore constructed a Hadamard state with energy density less than $-\xi\kappa^4\tau^4/(6\pi^2(\tau')^4)$ on an arbitrary region $B(\tau')$ (by translational invariance, the same applies to any other spacetime ball of radius τ'). The total expected energy of this state is $2\kappa\tau/(3\tau')$.

Note that the product of κ' and τ' is constant. This shows that we may arrange for large regions of negative energy density albeit with low magnitude. We may extend the example as follows. Suppose a constant energy density $\rho_0 > 0$ is given, and choose an integer $j > 6\pi^2\rho_0/(\xi(\kappa')^4)$. Then the j-particle state $\Psi_{\kappa'}^{\otimes j} = \Psi_{\kappa'}\otimes\cdots\otimes\Psi_{\kappa'}$ has energy density

$$\left\langle\Psi_{\kappa'}^{\otimes j}\middle|\rho^{\text{quant}}\Psi_{\kappa'}^{\otimes j}\right\rangle(x) = j\langle\Psi_{\kappa'}|\rho^{\text{quant}}\Psi_{\kappa'}\rangle(x) < -\rho_0 \qquad \text{for all} \quad x \in B(\tau'), \tag{33}$$

[4] The well-known identity $\int_0^\infty \text{d}k\,k^n\,\text{e}^{-k} = n!$ makes most of the calculations almost trivial.

and total energy

$$\left\langle \Psi_{\kappa'}^{\otimes j} \middle| H \Psi_{\kappa'}^{\otimes j} \right\rangle = \frac{2j\kappa'}{3} > \frac{2\pi^2\rho_0}{\xi(\kappa')^3} = (\tau')^3 \frac{2\pi^2\rho_0}{\xi\kappa^3\tau^3}, \tag{34}$$

illustrating that the large negative energy density effects also require large positive overall energy, at least in this example. We will see later that this is a general phenomenon.

Summarizing, we have shown that to any bounded subset $\mathcal{O}$ of Minkowski space and arbitrary constant $\rho_0 > 0$ there is a Hadamard state in which the expected energy density is less than $-\rho_0$ throughout $\mathcal{O}$. In particular, any smearing $\rho^{\text{quant}}(\mathfrak{f})$ with a non-negative compactly supported distribution $\mathfrak{f}$ is unbounded from below on the class of Hadamard states.

4. Quantum energy inequalities

In this section, we are going to derive the main result that is to give a lower bound for time-like averages of the energy density. We start with a quantum field on a curved spacetime. In a second step, we specialize these results to Minkowski space, where the vacuum state is the preferred reference state.

4.1. Globally hyperbolic spacetime

We keep the same assumptions as in section 2; in particular, γ is a time-like, connected geodesic parameterized by proper time. Our goal is to find a lower bound for the weighted average of the quantum energy density on γ,

$$\langle \rho^{\text{quant}} \circ \gamma \rangle_\Psi(\mathfrak{f}) = \int \mathrm{d}\tau\, \mathfrak{f}(\tau) \langle \rho^{\text{quant}} \rangle_\Psi(\gamma(\tau)) \tag{35}$$

in the case where $\mathfrak{f} = f^2$ for some real-valued function $f \in \mathscr{D}(\mathbb{R}, \mathbb{R})$. The main task is to rewrite (35) in such a way that the lower bound may be deduced by discarding manifestly positive terms.

To start, we use (14) to get

$$\begin{aligned} \langle \rho^{\text{quant}} \circ \gamma \rangle_\Psi(f^2) = {} & \left(\left[\hat{\rho}_1 {:}\omega_2^\Psi{:} \right]_c \circ \gamma \right)(f^2) - \xi \left(\left[\hat{\rho}_2 {:}\omega_2^\Psi{:} \right]_c \circ \gamma \right)(f^2) \\ & + \xi \left(\left[\hat{\rho}_3 {:}\omega_2^\Psi{:} \right]_c \circ \gamma \right)(f^2). \end{aligned} \tag{36}$$

Each term on the right-hand side will be treated in turn. It will also be useful to define $\varphi(\tau, \tau') = (\gamma(\tau), \gamma(\tau'))$, and to write $\varphi^* F$ to denote the pullback $\varphi^* F(\tau, \tau') = F(\gamma(\tau), \gamma(\tau'))$ of a smooth function F from $M \times M$ to $\mathbb{R} \times \mathbb{R}$.

The first term on the right-hand side in (36) may then be rewritten, following [5], as

$$\begin{aligned} \left(\left[\hat{\rho}_1 {:}\omega_2^\Psi{:} \right]_c \circ \gamma \right)(f^2) &= \int \mathrm{d}\tau\, f^2(\tau) \varphi^* \left(\hat{\rho}_1 {:}\omega_2^\Psi{:} \right)(\tau, \tau) \\ &= \int \mathrm{d}\tau\, \mathrm{d}\tau' \delta(\tau - \tau') f(\tau) f(\tau') \varphi^* \left(\hat{\rho}_1 {:}\omega_2^\Psi{:} \right)(\tau, \tau') \\ &= \int_0^\infty \frac{\mathrm{d}\alpha}{\pi} \int \mathrm{d}\tau\, \mathrm{d}\tau'\, \mathrm{e}^{-\mathrm{i}\alpha(\tau - \tau')} f(\tau) f(\tau') \varphi^* \left(\hat{\rho}_1 {:}\omega_2^\Psi{:} \right)(\tau, \tau') \\ &= \int_0^\infty \frac{\mathrm{d}\alpha}{\pi} \varphi^* \left(\hat{\rho}_1 {:}\omega_2^\Psi{:} \right)(\overline{f_\alpha}, f_\alpha) \\ &= \int_0^\infty \frac{\mathrm{d}\alpha}{\pi} \varphi^* \left(\hat{\rho}_1 \omega_2^\Psi \right)(\overline{f_\alpha}, f_\alpha) - \int_0^\infty \frac{\mathrm{d}\alpha}{\pi} \varphi^* \left(\hat{\rho}_1 \omega_2^0 \right)(\overline{f_\alpha}, f_\alpha), \end{aligned} \tag{37}$$

where $f_\alpha(\tau) = \mathrm{e}^{\mathrm{i}\alpha\tau} f(\tau)$. Here, we have made use of the Fourier representation of the δ-function and also the symmetry of the normal ordered two-point function to arrange that the α-integral takes place over the positive half-axis. A decomposition of this kind will be referred to as a *point-splitting trick*, see [5]. The distributional pullbacks of the form $\varphi^*(\hat{\rho}_1\omega_2^\Psi)$ appearing in the last step were shown to exist in [5], using the microlocal characterization of Hadamard states given in [13]. Moreover, if $\xi \leqslant 1/4$ then these distributions are positive type, i.e., $\varphi^*(\hat{\rho}_1\omega_2^\Psi)(\overline{\zeta}, \zeta) \geqslant 0$ for all $\zeta \in \mathscr{D}(\mathbb{R})$. This is a direct consequence of theorem 2.2 in [5] and the form of $\hat{\rho}_1$. In particular, each integrand in the bottom line in (37) is non-negative. Finally, the integrals converge, because (as shown in [5]) $\varphi^*(\hat{\rho}\omega_2^\Psi)(\overline{f_\alpha}, f_\alpha)$ is of rapid decay as $\alpha \to +\infty$ for any Hadamard state Ψ and partial differential operator $\hat{\rho}$ with smooth coefficients.

To treat the second term on the right-hand side in (36), we will need the following identity, proved in the appendix.

Theorem 4.1. *Let F be a smooth function on $M \times M$ and ∂ be a partial differential operator of the form $\partial = \zeta^\mu \nabla_\mu$, with a smooth vector field ζ. Then*

$$2h^2[(\mathbb{1} \otimes_\mathfrak{s} \partial^2)F]_c + \partial[(\mathbb{1} \otimes_\mathfrak{s} \partial h^2)F - (\mathbb{1} \otimes_\mathfrak{s} h^2\partial)F]_c = -2[(\partial h \otimes_\mathfrak{s} \partial h)F]_c + 2(\partial h)^2[(\mathbb{1} \otimes_\mathfrak{s} \mathbb{1})F]_c + 2\partial[(h \otimes_\mathfrak{s} \partial h)F]_c, \tag{38}$$

for $h \in \mathscr{D}(M, \mathbb{R})$.

Now choose a function $f_\mathcal{T} \in \mathscr{D}(\mathcal{T}, \mathbb{R})$ such that $f_\mathcal{T} \circ \gamma = f$. Applying theorem 4.1 with $F = :\omega_2^\Psi:$, $h = f_\mathcal{T}$ and $\partial = \zeta^\mu\nabla_\mu$, where ζ is some smooth vector field on M with the property that $\zeta|_\gamma = \dot{\gamma}$, yields the identity

$$2f_\mathcal{T}^2\big[\big(\mathbb{1} \otimes_\mathfrak{s} \nabla_\zeta^2\big):\omega_2^\Psi:\big]_c + \nabla_\zeta\big[\big(\mathbb{1} \otimes_\mathfrak{s} \nabla_\zeta f_\mathcal{T}^2\big):\omega_2^\Psi: - \big(\mathbb{1} \otimes_\mathfrak{s} f_\mathcal{T}^2\nabla_\zeta\big):\omega_2^\Psi:\big]_c = -2\big[(\nabla_\zeta f_\mathcal{T} \otimes_\mathfrak{s} \nabla_\zeta f_\mathcal{T}):\omega_2^\Psi:\big]_c + 2f_\mathcal{T}^2\nabla_\zeta\big[:\omega_2^\Psi:\big]_c + 2\nabla_\zeta\big[(f_\mathcal{T} \otimes_\mathfrak{s} \nabla_\zeta f_\mathcal{T}):\omega_2^\Psi:\big]_c. \tag{39}$$

All expressions are well defined, since $:\omega_2^\Psi:$ is smooth. The first expression on the left-hand side is nothing but $f_\mathcal{T}^2\big[\hat{\rho}_2:\omega_2^\Psi:\big]_c$. Let us turn to the other terms. Recalling that τ is the proper time along γ, we can write

$$\int \mathrm{d}\tau\big(\nabla_{\dot{\gamma}}\big[\big(\mathbb{1} \otimes_\mathfrak{s} \nabla_{\dot{\gamma}} f_\mathcal{T}^2\big):\omega_2^\Psi:\big]_c \circ \gamma\big)(\tau) = \int \mathrm{d}\tau\, \partial_\tau\big(\big[\big(\mathbb{1} \otimes_\mathfrak{s} \nabla_{\dot{\gamma}} f_\mathcal{T}^2\big):\omega_2^\Psi:\big]_c \circ \gamma\big)(\tau), \tag{40}$$

which vanishes, since $f_\mathcal{T}$ is of compact support. Further terms in (39) will vanish for the same reasons after integration. We finally obtain that

$$\int \mathrm{d}\tau\, f^2(\tau)\big(\big[\hat{\rho}_2:\omega_2^\Psi:\big]_c \circ \gamma\big)(\tau) = 2\int \mathrm{d}\tau\,(\partial_\tau f)^2\varphi^*:\omega_2^\Psi:(\tau, \tau) - 2\int \mathrm{d}\tau\big(\big[(\nabla_{\dot{\gamma}} f_\mathcal{T} \otimes_\mathfrak{s} \nabla_{\dot{\gamma}} f_\mathcal{T}):\omega_2^\Psi:\big]_c \circ \gamma\big)(\tau). \tag{41}$$

The second integral on the right-hand side may be rewritten (up to a factor), using the point-splitting trick, as

$$\begin{aligned}\int \mathrm{d}\tau\big(\big[(\nabla_{\dot{\gamma}} f_\mathcal{T} \otimes_\mathfrak{s} \nabla_{\dot{\gamma}} f_\mathcal{T}):\omega_2^\Psi:\big]_c \circ \gamma\big)(\tau) &= \int_0^\infty \frac{\mathrm{d}\alpha}{\pi}\int \mathrm{d}\tau\,\mathrm{d}\tau'\,\mathrm{e}^{-\mathrm{i}\alpha(\tau-\tau')}\partial_\tau\partial_{\tau'}\big(f(\tau)f(\tau')\varphi^*:\omega_2^\Psi:(\tau, \tau')\big)\\ &= \int_0^\infty \frac{\mathrm{d}\alpha}{\pi}\alpha^2\int \mathrm{d}\tau\,\mathrm{d}\tau'\,\overline{f_\alpha}(\tau)f_\alpha(\tau')\varphi^*:\omega_2^\Psi:(\tau, \tau')\\ &= \int_0^\infty \frac{\mathrm{d}\alpha}{\pi}\alpha^2\varphi^*\omega_2^\Psi(\overline{f_\alpha}, f_\alpha) - \int_0^\infty \frac{\mathrm{d}\alpha}{\pi}\alpha^2\varphi^*\omega_2^0(\overline{f_\alpha}, f_\alpha),\end{aligned} \tag{42}$$

where we have also used integration by parts in τ, τ' and the fact that f is of compact support. As before, the bottom line of (42) is a difference of two non-negative terms.

Finally, let us put the results of (37), (41) and (42) together with the remaining term in (36).[5] We find that

$$\begin{aligned}\langle\rho^{\text{quant}}\circ\gamma\rangle_\Psi(f^2) &= \int_0^\infty \frac{\mathrm{d}\alpha}{\pi}\varphi^*\big(\hat\rho_1\omega_2^\Psi\big)(\overline{f_\alpha}, f_\alpha) - \int_0^\infty \frac{\mathrm{d}\alpha}{\pi}\varphi^*\big(\hat\rho_1\omega_2^0\big)(\overline{f_\alpha}, f_\alpha)\\ &\quad + 2\xi\int_0^\infty \frac{\mathrm{d}\alpha}{\pi}\alpha^2\varphi^*\omega_2^\Psi(\overline{f_\alpha}, f_\alpha) - 2\xi\int_0^\infty \frac{\mathrm{d}\alpha}{\pi}\alpha^2\varphi^*\omega_2^0(\overline{f_\alpha}, f_\alpha)\\ &\quad - 2\xi\int \mathrm{d}\tau\,(\partial_\tau f)^2\varphi^*{:}\omega_2^\Psi{:}(\tau,\tau) + \xi\int \mathrm{d}\tau\, f^2(\tau)\varphi^*\big(\hat\rho_3{:}\omega_2^\Psi{:}\big)(\tau,\tau), \end{aligned} \tag{43}$$

which, noting that $\varphi^*{:}\omega_2^\Psi{:}(\tau,\tau) = \langle{:}\Phi^2{:}\circ\gamma\rangle_\Psi(\tau)$, results in the following theorem.

Theorem 4.2. *Let ω_2^0 be the two-point function of a reference Hadamard state for the non-minimally coupled scalar field with coupling constant $\xi \in [0, 1/4]$, defined on a globally hyperbolic spacetime with smooth metric. Furthermore, let γ be a time-like geodesic parameterized in proper time τ and let $f \in \mathscr{D}(\mathbb{R},\mathbb{R})$. On the set of Hadamard states, we then find*

$$(\rho^{\text{quant}}\circ\gamma)(f^2) \geqslant -\mathfrak{Q}^\xi(f), \tag{44}$$

where

$$\mathfrak{Q}^\xi(f) = \widetilde{\mathfrak{Q}}_A^\xi(f)\mathbb{1} + \xi({:}\Phi^2{:}\circ\gamma)(\mathfrak{Q}_B[f]) + \xi({:}\Phi^2{:}\circ\gamma)\big(\mathfrak{Q}_C^\xi[f]\big), \tag{45}$$

with

$$\widetilde{\mathfrak{Q}}_A^\xi(f) = \int_0^\infty \frac{\mathrm{d}\alpha}{\pi}\big[\varphi^*\big(\hat\rho_1\omega_2^0\big)(\overline{f_\alpha}, f_\alpha) + 2\xi\alpha^2\varphi^*\omega_2^0(\overline{f_\alpha}, f_\alpha)\big], \tag{46}$$

and $\mathfrak{Q}_B(f)$ and $\mathfrak{Q}_C^\xi(f)$ are functions in $\mathscr{D}(\mathbb{R},\mathbb{R})$ given by

$$\mathfrak{Q}_B[f](\tau) = 2(\partial_\tau f(\tau))^2, \tag{47}$$

$$\mathfrak{Q}_C^\xi[f](\tau) = f(\tau)^2\big(R_{\mu\nu}\gamma^\mu\gamma^\nu - \tfrac{1}{2}(1-4\xi)R\big)(\tau). \tag{48}$$

Furthermore, $\widetilde{\mathfrak{Q}}_A^\xi(f)$ and $\mathfrak{Q}_B[f]$ are non-negative.

This result follows from the previous discussion by discarding manifestly positive terms. We remark that $\widetilde{\mathfrak{Q}}_A^\xi(f)$ depends on the reference state and that for $\xi = 0$, we recover results known for minimal coupling [5]. Moreover, $\mathfrak{Q}_C^\xi(f)$ vanishes if the region of interest is Ricci-flat.

4.2. Minkowski space

In this section, we apply the results derived in the previous subsection to n-dimensional Minkowski space. Without loss of generality, we average in the time argument $\tau = t$ at the spatial origin, i.e., $\gamma(\tau) = (\tau, \mathbf{x}_0)$. We choose our reference state to be the vacuum state Ω, which has a two-point function

$$\omega_2^\Omega(t,\mathbf{x},t',\mathbf{x}') = \int \mathrm{d}\mu(\mathbf{k})\,\mathrm{e}^{-\mathrm{i}[(t-t')\omega(\mathbf{k})-(\mathbf{x}-\mathbf{x}')\mathbf{k}]}, \tag{49}$$

[5] We do not apply the point-splitting trick to this term as we cannot necessarily find smooth real square roots of the geometrical quantities involved.

in the distributional sense. For $g \in \mathscr{D}(\mathbb{R})$, we find that

$$\varphi^* \omega_2^{\Omega}(\overline{g} \otimes g) = \frac{1}{2} \frac{S_{n-2}}{(2\pi)^{n-1}} \int_0^\infty \mathrm{d}k \frac{k^{n-2}}{\omega(k)} |\hat{g}(\omega(k))|^2, \tag{50}$$

where S_{n-2} is the surface area of the $(n-2)$-dimensional standard unit sphere[6]. We also have the identity

$$m^2 \varphi^* \omega_2^{\Omega}(\overline{g} \otimes g) + \sum_{i=1}^{n-1} \varphi^*\big((\partial_i \otimes \partial_i)\omega_2^{\Omega}\big)(\overline{g} \otimes g) = \varphi^*\big((\partial_0 \otimes \partial_0)\omega_2^{\Omega}\big)(\overline{g} \otimes g), \tag{51}$$

which follows from the spacetime translation invariance of the vacuum and the field equation (5). So one can absorb the mass term and the spatial derivatives appearing in the definition of $\widetilde{\mathfrak{Q}}_A^{\xi}(f)$ (via $\hat{\rho}_1$) into a further term that involves time derivatives. One finds that (44) becomes

Theorem 4.3. *For the non-minimally coupled scalar quantum field in n-dimensional Minkowski space* $\mathbf{M}^n_{\mathrm{Mink}}$,

$$(\rho^{\mathrm{quant}} \circ \gamma)(f^2) \geqslant -\mathfrak{Q}^{\xi}(f) = -\big(\widetilde{\mathfrak{Q}}_A^{\xi}(f)\mathbb{1} + \xi(:\Phi^2: \circ \gamma)(\mathfrak{Q}_B[f])\big), \tag{52}$$

in the sense of quadratic forms on Hadamard states, where

$$\widetilde{\mathfrak{Q}}_A^{\xi}(f) = \frac{S_{n-2}}{(2\pi)^n} \int_0^\infty \mathrm{d}\alpha \int_0^\infty \mathrm{d}k \frac{k^{n-2}}{\omega(k)} ((1-2\xi)\omega^2(k) + 2\xi\alpha^2)|\hat{f}(\alpha + \omega(k))|^2 \tag{53}$$

and

$$\mathfrak{Q}_B[f](t) = 2(\partial_t f(t))^2 \tag{54}$$

for $f \in \mathscr{D}(\mathbb{R}, \mathbb{R})$ *and* $\xi \in [0, 1/4]$.

It is easy to see that $\widetilde{\mathfrak{Q}}_A^{\xi}(f)$ is non-negative for $\xi \in [0, 1/4]$ and that $\widetilde{\mathfrak{Q}}_A^{\xi=0}(f) = \mathfrak{Q}(f)$, where $\mathfrak{Q}(f)$ is the lower bound found in [16] for the minimally coupled ($\xi = 0$) scalar field. As a consequence, theorem 4.3 recovers the results of [16] for minimal coupling.

We can write $\widetilde{\mathfrak{Q}}_A^{\xi}(f)$ in the form[7]

$$\widetilde{\mathfrak{Q}}_A^{\xi}(f) = \frac{S_{n-2}}{(2\pi)^n} \int_m^\infty \mathrm{d}u |\hat{f}|^2(u) u^n \times \left(\frac{1}{n} Q_{n,2}\left(\frac{u}{m}\right) - 4\xi \frac{1}{n-1} Q_{n,1}\left(\frac{u}{m}\right) + 2\xi \frac{1}{n-2} Q_{n,0}\left(\frac{u}{m}\right)\right), \tag{55}$$

where the non-negative functions $Q_{n,k}$ are defined by

$$Q_{n,k}(y) = \frac{n+k-2}{y^{n+k-2}} \int_1^y \mathrm{d}x (x^2-1)^{(n-3)/2} x^k, \tag{56}$$

for $n + k \geqslant 2$. They vanish for $k + n = 2$ and else have the properties that $Q_{n,k}(1) = 0$ and $Q_{n,k}(y) \to 1$ as $y \to \infty$. As they are continuous, they are therefore bounded. It might be useful to use the following estimate for $\widetilde{\mathfrak{Q}}_A^{\xi}(f)$, which follows from the previous discussion for $n > 2$ and $\xi \in [0, 1/4]$,

$$\widetilde{\mathfrak{Q}}_A^{\xi}(f) \leqslant \frac{S_{n-2}}{(2\pi)^n} \frac{3n-4}{2n(n-2)} \int_0^\infty \mathrm{d}u |\hat{f}|^2(u) u^n. \tag{57}$$

This estimate is true for the massive and the massless case and one can find a similar estimate for the two-dimensional case.

[6] We have $S_m = 2\sqrt{\pi^m}/\Gamma(m/2)$, where Γ is the Gamma function.

[7] We will assume that $n > 2$ and $m > 0$, but one can find similar expressions for these cases as well.

To conclude this subsection, we investigate the behaviour under rescaling of the averaging function. The smearing function $f_\lambda(t) = f(t/\lambda)/\sqrt{\lambda}$, for $\lambda > 0$, has the property that $\|f_\lambda\|_{L^2} = \|f\|_{L^2}$, and its Fourier transform satisfies the identity

$$(\hat{f}_\lambda(u))^2 = \lambda(\hat{f}(\lambda u))^2. \tag{58}$$

One can conclude from this that $\widetilde{\mathfrak{Q}}_A^\xi(f_\lambda) = O(\lambda^{-n})$ as $\lambda \to \infty$. In fact, even faster decay could be concluded if $m > 0$ using the arguments of [17]. It follows that $\lambda\widetilde{\mathfrak{Q}}_A^\xi(f_\lambda) \to 0$ as $\lambda \to \infty$. For states with $|\langle{:}\Phi^2{:} \circ \gamma\rangle_\Psi(t)| < c(1+|t|)^{1-\varepsilon}$, for some positive constants c, ε one can show that $\lambda\langle{:}\Phi^2{:} \circ \gamma\rangle_\Psi(\mathfrak{Q}_B(f_\lambda)) \to 0$ as $\lambda \to \infty$, and we therefore obtain the averaged weak energy condition (AWEC) for $\xi \in [0, 1/4]$ in the form

$$\liminf_{\lambda\to\infty} \int \mathrm{d}t\,\langle\rho^{\text{quant}} \circ \gamma\rangle_\Psi(t) f(t/\lambda)^2 \geqslant 0. \tag{59}$$

This is in line with a result in [18], which shows that AWEC holds for states in which the particle number and the energy is bounded (see the penultimate paragraph in section III of [18]). In the minimally coupled case, AWEC is also known to follow from QEIs [16, 19]. We will return to the AWEC briefly in section 5.2.

5. Investigation of state dependence

The lower bounds that we derived have the characteristic that they are state-dependent except for minimal coupling. As all previously known QEIs are state-independent, it is important to understand the nature of the state dependence to ensure that our bounds are not vacuous.

5.1. KMS states and temperature scaling

In this subsection, we analyse the temperature scaling behaviour of the stress–energy tensor and the bound in theorem 4.3 for a KMS state Ψ^β, i.e., a thermal equilibrium state at positive temperature β^{-1} as seen by the observer on γ. Its two-point function ω_2^β in a n-dimensional Minkowski space, with $n > 3$, is given by

$$\omega_2^\beta(t, \mathbf{x}, t', \mathbf{x}') = \int \mathrm{d}\mu(\mathbf{k}) \left(\frac{\mathrm{e}^{-\mathrm{i}((t-t')\omega(\mathbf{k}) - (\mathbf{x}-\mathbf{x}')\mathbf{k})}}{1 - \mathrm{e}^{-\beta\omega(\mathbf{k})}} + \frac{\mathrm{e}^{+\mathrm{i}((t-t')\omega(\mathbf{k}) - (\mathbf{x}-\mathbf{x}')\mathbf{k})}}{\mathrm{e}^{\beta\omega(\mathbf{k})} - 1} \right). \tag{60}$$

We renormalize the two-point function of the KMS-state by subtracting the two-point function of the vacuum (49). In the coincidence limit, we find that

$$\begin{aligned} [{:}\omega_2^\beta{:}]_c(t, \mathbf{x}) &= \int \mathrm{d}\mu(\mathbf{k}) \frac{2}{\mathrm{e}^{\beta\omega(\mathbf{k})} - 1} \\ &= B_{n,0}(\beta m), \end{aligned} \tag{61}$$

with the positive function $B_{n,k}$ defined on $[0, \infty)$ for $k \geqslant 0$ by

$$B_{n,k}(\alpha) = \frac{S_{n-2}}{(2\pi)^{n-1}} \int_\alpha^\infty \mathrm{d}z (z^2 - \alpha^2)^{\frac{n-3}{2}} \frac{z^k}{\mathrm{e}^z - 1}. \tag{62}$$

Expression (61) is positive and invariant under spacetime translations. Thus, the state-dependent part of the lower bound in theorem 4.3 is given by

$$\langle{:}\Phi^2{:} \circ \gamma\rangle_{\Psi^\beta}(\mathfrak{Q}_B[f]) = 2\beta^{2-n} B_{n,0}(\beta m) \|f'\|_{L^2}^2, \tag{63}$$

while the state-independent part $\widetilde{\mathfrak{Q}}^{\xi}(f)$ is obviously independent of β. On the other hand, the renormalized energy density of this state is

$$\begin{aligned}\langle\rho^{\text{quant}}\rangle_{\Psi^\beta}(t,\mathbf{x}) &= \left[\hat{\rho}{:}\omega_2^\beta{:}\right]_c(t,\mathbf{x})\\ &= \int \mathrm{d}\mu(\mathbf{k})\frac{2\omega^2(\mathbf{k})}{\mathrm{e}^{\beta\omega(\mathbf{k})}-1}\\ &= \beta^{-n}B_{n,2}(\beta m),\end{aligned} \tag{64}$$

so that the time-averaged energy density is given by

$$\langle\rho^{\text{quant}}\circ\gamma\rangle_{\Psi^\beta}(f^2) = \beta^{-n}B_{n,2}(\beta m)\|f\|_{L^2}^2. \tag{65}$$

We can now state the non-triviality result.

Theorem 5.1. *The bound for the energy density of a non-minimally coupled scalar quantum field given in theorem 4.3 is non-trivial in the sense of [9], i.e., there do not exist constants c, c' such that*

$$|\langle\rho^{\text{quant}}\circ\gamma\rangle_\Psi(f^2)| \leqslant c + c'|\langle\mathfrak{Q}^\xi(f)\rangle_\Psi| \tag{66}$$

for all Hadamard states Ψ unless f is identically zero.

Proof. From the previous discussion, we find that for a fixed non-trivial smearing function f, in the limit of high temperatures

$$\lim_{\beta\to 0}\beta^n\langle\rho^{\text{quant}}\circ\gamma\rangle_{\Psi^\beta}(f^2) = B_{n,2}(0)\|f\|_{L^2}^2 > 0, \tag{67}$$

and

$$\lim_{\beta\to 0}\beta^n\widetilde{\mathfrak{Q}}_A^\xi(f) = \lim_{\beta\to 0}\beta^n\langle{:}\Phi^2{:}\circ\gamma\rangle_{\Psi^\beta}(\mathfrak{Q}_B[f]) = 0. \tag{68}$$

Now if the bound in theorem 4.3 is trivial, then there exists some constant c such that (4) holds. This implies that

$$0 < B_{n,2}(0)\|f\|_{L^2}^2 \leqslant \lim_{\beta\to 0}\beta^n(c + c'|\widetilde{\mathfrak{Q}}_A^\xi(f)| + c'|\xi\langle{:}\Phi^2{:}\circ\gamma\rangle_{\Psi^\beta}(\mathfrak{Q}_B[f])|) = 0, \tag{69}$$

which is a contradiction. □

A more refined formulation of this result would be that our bound is *non-trivial with respect to high temperature scaling for KMS states*.

5.2. Energy behaviour

The previous results already show that the lower bound in theorem 4.3 has a different scaling behaviour from the energy density itself. We will now present a more general analysis that gives more insight into this, again working in four-dimensional Minkowski space $\mathbf{M}^4_{\text{Mink}}$.

The lower bound in theorem 4.3 only depends on the fields ${:}\Phi^2{:}$ and $\mathbb{1}$. On the other hand, the energy density also involves terms such as ${:}\dot{\Phi}^2{:}$. This results in a crucial difference in their energy behaviour. To be more precise, we seek values of $p, q \in \mathbb{R}^+$ for which there are constants $c_{f,q}, c'_{f,p}$ such that

$$c_{f,q}(H + m\mathbb{1})^q \geqslant (\rho^{\text{quant}}\circ\gamma)(f^2) \geqslant -c'_{f,p}(H + m\mathbb{1})^p \tag{70}$$

holds (in the sense of quadratic forms) on the set of Hadamard vector states. In the minimally coupled case, we know that there is a state-independent lower bound so we may take $p = 0$; however, for $\xi \in (0, 1/4]$ we have already shown that $(\rho^{\text{quant}}\circ\gamma)(f^2)$ is unbounded from below, so p must be strictly positive if (70) is to hold. By theorem 4.3, it is enough to show that $\mathfrak{Q}^{\xi}(f) \leqslant c'_{f,p}(H + m\mathbb{1})^p$ to conclude that the right-hand inequality in (70) holds; we will show that this is possible for any $p > 2$.

On the other hand, we will show that the left-hand inequality in (70) cannot be satisfied for $q < 3$. In this sense, our lower bound represents a non-trivial constraint. Indeed, the situation here is reminiscent of the sharp Gårding inequalities studied in the theory of pseudodifferential operators, in which operators with positive classical symbols may be bounded from below 'with a gain in derivatives', i.e., by an operator of lower order. Although the analogy is not direct, it seems worthy of further investigation.

As a consequence of this analysis we immediately obtain another proof of theorem 5.1, namely that the lower bound in theorem 4.3 is non-trivial in the sense of [9] (but this time using states in the domain of a power of the Hamiltonian rather than KMS states).

We begin by establishing our claim relating to the left-hand inequality in (70). To do this we first consider the massless field and construct the following one-particle state:

$$\Psi_\kappa = \frac{4\pi}{\kappa}\int \mathrm{d}\mu(\mathbf{k})\,\mathrm{e}^{-|\mathbf{k}|/\kappa} a^\dagger(\mathbf{k})\Omega. \tag{71}$$

It is straightforward to calculate that $\langle H^j\rangle_{\Psi_\kappa} = \left(\frac{\kappa}{2}\right)^j (j+1)!$ and that[8]

$$\langle \rho^{\text{quant}} \circ \gamma\rangle_{\Psi_\kappa}(t) = \frac{\kappa^4}{\pi^2}\left\{\frac{4}{1+t^2\kappa^2} - 4\xi\frac{6(t^2\kappa^2-1)}{(1+t^2\kappa^2)^4}\right\}. \tag{72}$$

We have a peak at $t = 0$, where the expectation value scales like the fourth power in the Hamiltonian. However, this pointwise behaviour will not hold for the smeared field. Let us assume that f is a non-trivial, integrable function. We find that

$$\lim_{\kappa\to\infty}\frac{\langle \rho^{\text{quant}} \circ \gamma\rangle_{\Psi_\kappa}(f^2)}{\langle H^3\rangle_{\Psi_\kappa}\|f\|^2_{L^2}} = \lim_{\kappa\to\infty}\frac{2\kappa^3(2+3\xi)/\pi}{3\kappa^3} = \frac{2(2+3\xi)}{3\pi}, \tag{73}$$

i.e., we have an asymptotic scaling like the expectation value of H^3. In the limit where κ becomes large, the high momenta are dominant. If the field is massive, the mass would therefore become negligible. We can deduce that the same scaling behaviour remains true for the massive field, at least asymptotically. So the smeared energy density for the scalar field in Minkowski space scales asymptotically at least with the third power of the Hamiltonian, i.e., the left-hand inequality in (70) can only hold if $q \geqslant 3$. It is worth remarking that we could have conducted the same analysis using the one-particle states investigated in section 3, and with the same result: although the energy density in these states is negative at the spacetime origin, one may check that the smeared energy density is positive for sufficiently large κ.

The second part of the discussion aims to establish suitable H-bounds on the state-dependent part of our lower bound, i.e., the field $:\Phi^2:$. H-bounds have been discussed elsewhere (see, e.g., [20]), but we require more detailed information on the power of the Hamiltonian involved and on the controlling constants than we have been able to locate in the literature. The following discussion may therefore be of independent interest.

We will now assume that $m > 0$, but allow general spacetime dimensions $n \geqslant 2$. Let h be the one-particle Hamiltonian and $\mathrm{d}\Gamma$ be the second quantization map (so, for example, the Hamiltonian is $H = \mathrm{d}\Gamma(h)$). Let us initially restrict to the domain $D_{\mathscr{S}} \subset \mathcal{F}_s(\mathcal{H})$, defined as the space of vectors in Fock space all of whose n-particle wavefunctions are Schwartz functions and all but finitely many of which vanish identically, see section X.7 in [15]. This is a dense domain in the Fock space and for every vector $\Psi \in D_{\mathscr{S}}$, we find that

8 For the concrete calculation in (72), we assumed for simplicity that the geodesic γ is located at the spatial origin.

$\mathbf{k} \mapsto \|a(\mathbf{k})\Psi\|^2 \in \mathscr{S}(\mathbb{R}^{n-1})$ and that

$$\|\mathrm{d}\Gamma(h^p)^{1/2}\Psi\|^2 = \int \mathrm{d}\mu(\mathbf{k})\omega^p(\mathbf{k})\|a(\mathbf{k})\Psi\|^2 \tag{74}$$

for any $p \in \mathbb{R}$.

Let $g \in \mathscr{S}(\mathbb{R}^{n-1}) \subset \mathcal{H}$. Then $\omega^{-p/2}g \in \mathcal{H}$, where ω acts on $\mathcal{H}$ by multiplication and the Cauchy–Schwarz inequality implies that

$$\begin{aligned}\|a(g)\Psi\|^2 &\leqslant \int \mathrm{d}\mu(\mathbf{k})\mathrm{d}\mu(\mathbf{k}')|\langle a(\mathbf{k})\Psi|a(\mathbf{k}')\Psi\rangle||g(\mathbf{k})||g(\mathbf{k}')| \\ &\leqslant \left\{\int \mathrm{d}\mu(\mathbf{k})\|a(\mathbf{k})\Psi\||g(\mathbf{k})|\right\}^2 \\ &= \left\{\int \mathrm{d}\mu(\mathbf{k})(\omega^{p/2}(\mathbf{k})\|a(\mathbf{k})\Psi\|)|\omega^{-p/2}g(\mathbf{k})|\right\}^2 \\ &\leqslant \|\mathrm{d}\Gamma(h^p)^{1/2}\Psi\|^2 \cdot \|\omega^{-p/2}g\|^2_{\mathcal{H}} \qquad p \in \mathbb{R},\end{aligned} \tag{75}$$

for $\Psi \in D_{\mathscr{S}}$. Due to the construction of $D_{\mathscr{S}}$, one can find that $D_{\mathscr{S}} \subset D(H^p)$ for any $p \in \mathbb{R}$. Now let $\Psi^{(l)} \subset D_{\mathscr{S}}$ be a l-particle state. We have

$$(H^p\Psi^{(l)})(\mathbf{k}_1, \ldots, \mathbf{k}_l) = \left(\sum_{i=1}^{l} \omega(\mathbf{k}_i)\right)^p \Psi^{(l)}(\mathbf{k}_1, \ldots, \mathbf{k}_l), \tag{76}$$

$$(\mathrm{d}\Gamma(h^p)\Psi^{(l)})(\mathbf{k}_1, \ldots, \mathbf{k}_l) = \sum_{i=1}^{l} \omega^p(\mathbf{k}_i)\Psi^{(l)}(\mathbf{k}_1, \ldots, \mathbf{k}_l). \tag{77}$$

For $a, b \geqslant 0$ and $p \geqslant 1$ we find that $(a+b)^p \geqslant a^p + b^p$,[9] so

$$\left(\sum_{i=1}^{l} \omega(\mathbf{k}_i)\right)^p \geqslant \sum_{i=1}^{l} \omega^p(\mathbf{k}_i). \tag{79}$$

This implies that $0 \leqslant \mathrm{d}\Gamma(h^p) \leqslant H^p$ on $D_{\mathscr{S}}$ for $p \geqslant 1$. Using this, the commutation relations and (75), we recover H-bounds of the form

$$\begin{aligned}\|a(g)\Psi\|^2 &\leqslant \|H^{p/2}\Psi\|^2 \cdot \|\omega^{-p/2}g\|^2_{\mathcal{H}}, \\ \|a^\dagger(g)\Psi\|^2 &\leqslant \|H^{p/2}\Psi\|^2 \cdot \|\omega^{-p/2}g\|^2_{\mathcal{H}} + \|g\|^2_{\mathcal{H}}\|\Psi\|^2,\end{aligned} \tag{80}$$

for $p \geqslant 1$ on $D_{\mathscr{S}}$.

Now define the distribution space,

$$\mathcal{L}_q = \left\{F \in \mathscr{E}'\left(\mathbf{M}^n_{\mathrm{Mink}}\right) | \|F\|^2_q := \|\omega^q \widetilde{F}\|^2_{\mathcal{H}} < \infty\right\}, \tag{81}$$

where $\|\cdot\|_q$ is a semi-norm. Since the field is massive we have the inclusion $\mathcal{L}_0 \subset \mathcal{L}_q$ for $q \leqslant 0$. The field $\Phi(F)$ defines an operator on Fock space on the domain $D((N+1)^{1/2})$, if $F, \overline{F} \in \mathcal{L}_0$ (or equivalently $\widetilde{F}, \widetilde{\overline{F}} \in \mathcal{H}$). Since $D_{\mathscr{S}} \subset D((N+1)^{1/2})$, $\Phi(F)$ is well-defined on $D_{\mathscr{S}}$.

[9] Since the function $x \to x^{p-1}$ is monotone increasing for $p \geqslant 1$, we have

$$(a+b)^p = a(a+b)^{p-1} + b(a+b)^{p-1} \geqslant a^p + b^p. \tag{78}$$

Now assume $p \geqslant 1$, $\Psi \in D_{\mathscr{S}}$ and $F, \overline{F} \in \mathcal{L}_0$. Using the inequality $(u+v)^2 \leqslant 2(u^2+v^2)$, we find

$$\|\Phi(F)\Psi\|^2 - \|\Phi(F)\Omega\|^2 \leqslant (\|a(\widetilde{\overline{F}})\Psi\| + \|a^\dagger(\widetilde{F})\Psi\|)^2 \leqslant 2\|H^{p/2}\Psi\|^2 \cdot \left(\|\overline{F}\|^2_{-p/2} + \|F\|^2_{-p/2}\right) + \|F\|^2_0\|\Psi\|^2. \tag{82}$$

Since this inequality is valid on $D_{\mathscr{S}}$, which was dense in the Fock space $\mathcal{F}_s(\mathcal{H})$, this result extends to all $\Psi \in D(H^{p/2})$.

In order to apply these results in conjunction with the point-splitting trick, we have to establish a connection between pulled back two-point functions and their representation in terms of distributionally smeared fields acting as operators on a certain domain. In particular, one has to check that for some $F(x) = \zeta(t) \otimes \delta_{\mathbf{x}_0}(\mathbf{x})$ with $\zeta \in \mathscr{D}(\mathbb{R})$, we have

$$\left(\varphi^*\omega_2^\Psi\right)(\overline{\zeta}, \zeta) = \|\Phi(F)\Psi\|^2. \tag{83}$$

The left-hand side is well-defined as discussed before. The right-hand side is well-defined as $F, \overline{F} \in \mathcal{L}_0$. The identity can then be shown by constructing a sequence $F_r \in \mathscr{D}\left(\mathbf{M}^n_{\text{Mink}}\right)$, with $\omega_2^\Psi(\overline{F_r}, F_r) \to \left(\varphi^*\omega_2^\Psi\right)(\overline{\zeta}, \zeta)$ and $\widetilde{F}_r \to \widetilde{F} = \widetilde{\zeta \otimes \delta_{\mathbf{x}_0}}$ in $\mathcal{H}$.[10] The latter property ensures that $\Phi(F_r)\Psi \to \Phi\left(\zeta \otimes \delta_{\mathbf{x}_0}\right)\Psi$ and the identity therefore holds because $\omega_2^\Psi(\overline{F_r}, F_r) = \|\Phi(F_r)\Psi\|^2$ for test functions $F_r \in \mathscr{D}\left(\mathbf{M}^n_{\text{Mink}}\right)$.

We are now able to apply the above result to find H-bounds on the Wick square, smeared along the inertial curve γ. Applying the point-splitting trick and (82), we find, for normalized Ψ, that

$$\begin{aligned}
\langle :\Phi^2: \circ\, \gamma\rangle_\Psi(f^2) &= \int_0^\infty \frac{\mathrm{d}\alpha}{\pi}\left(\|\Phi\left(f_\alpha \otimes \delta_{\mathbf{x}_0}\right)\Psi\|^2 - \|\Phi\left(f_\alpha \otimes \delta_{\mathbf{x}_0}\right)\Omega\|^2\right) \\
&\leqslant 2\|H^{p/2}\Psi\|^2 \cdot \int_0^\infty \frac{\mathrm{d}\alpha}{\pi}\left(\left\|\overline{f_\alpha \otimes \delta_{\mathbf{x}_0}}\right\|^2_{-p/2} + \left\|f_\alpha \otimes \delta_{\mathbf{x}_0}\right\|^2_{-p/2}\right) \\
&\quad + \|\Psi\|^2 \int_0^\infty \frac{\mathrm{d}\alpha}{\pi}\left\|f_\alpha \otimes \delta_{\mathbf{x}_0}\right\|^2_0 \\
&= 2\|H^{p/2}\Psi\|^2 \cdot \int_{-\infty}^\infty \frac{\mathrm{d}\alpha}{\pi}\left\|f_\alpha \otimes \delta_{\mathbf{x}_0}\right\|^2_{-p/2} + \|\Psi\|^2 \int_0^\infty \frac{\mathrm{d}\alpha}{\pi}\left\|f_\alpha \otimes \delta_{\mathbf{x}_0}\right\|^2_0,
\end{aligned} \tag{84}$$

where $f \in \mathscr{D}(\mathbb{R}, \mathbb{R})$ and we used (83) with $F(x) = f(t)\,\mathrm{e}^{\mathrm{i}\alpha t} \otimes \delta_{\mathbf{x}_0}(\mathbf{x})$. For convenience, let us introduce the positive quadratic functionals

$$B_{-p/2}(f) = 2\int_{-\infty}^\infty \frac{\mathrm{d}\alpha}{\pi}\left\|f_\alpha \otimes \delta_{\mathbf{x}_0}\right\|^2_{-p/2}, \tag{85}$$

$$C_0(f) = \int_0^\infty \frac{\mathrm{d}\alpha}{\pi}\left\|f_\alpha \otimes \delta_{\mathbf{x}_0}\right\|^2_0. \tag{86}$$

Lemma 5.2. *Let $(p+2) > n$, where n is the spacetime dimension, and let $f \in \mathscr{S}(\mathbb{R}, \mathbb{R})$. For the massive case, $C_0(f) < \infty$ and $B_{-p/2}(f) < \infty$.*

Proof. To show that $|C_0(f)| < \infty$, we see that

$$\left|\widetilde{f_\alpha \otimes \delta_{\mathbf{x}_0}}(\mathbf{k})\right|^2 = |\hat{f}(\omega(\mathbf{k}) + \alpha)|^2 \leqslant \frac{c}{(m + \omega(\mathbf{k}) + \alpha)^{n+1}} \leqslant \frac{1}{(m+\alpha)^2}\frac{c}{\omega^{n-1}(\mathbf{k})}, \tag{87}$$

[10] One can do this by defining $F_r(t, \mathbf{x}) = \zeta(t)\chi_r(\mathbf{x})$, with the approximate identity $\chi_r \in \mathscr{D}(\mathbb{R}^{n-1})$.

for some positive constant c, where we made use of the fact that f is in Schwartz space. It follows that there exists a constant c', such that

$$\| f_\alpha \otimes \delta_{\mathbf{x}_0} \|_0^2 \leqslant \frac{c'}{(m+\alpha)^2}, \tag{88}$$

which is integrable in α on $\mathbb{R}^+$ proving that $C_0(f) < \infty$.

To show that $|B_{-p/2}(f)| < \infty$, realize that there exists a constant c, such that

$$\left| \widetilde{f_\alpha \otimes \delta_{\mathbf{x}_0}}(\mathbf{k}) \right|^2 = |\hat{f}(\omega(\mathbf{k}) + \alpha)|^2 \leqslant \frac{c}{(m^2 + (\omega(\mathbf{k}) + \alpha)^2)}. \tag{89}$$

Using this inequality, we get

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{\pi} \| f_\alpha \otimes \delta_{\mathbf{x}_0} \|_{-p/2}^2 &\leqslant \int \mathrm{d}\mu(\mathbf{k}) \omega^{-p}(\mathbf{k}) \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{\pi} \frac{c}{(m^2 + (\omega(\mathbf{k}) + \alpha)^2)} \\ &< \frac{c}{m} \int \mathrm{d}\mu(\mathbf{k}) \omega^{-p}(\mathbf{k}), \end{aligned} \tag{90}$$

where we made use of Tonelli's theorem. The last expression, however, is finite due to the restriction on p. □

We note that, under the assumptions of lemma 5.2, the quantities in (85) and (86) may be estimated by similar arguments to those used to obtain (57). Combining inequality (84) and lemma 5.2, we can now summarize our result as the following H-bound:

Theorem 5.3. *Let n be the spacetime dimension, $p > (n-2)$, $f \in \mathscr{D}(\mathbb{R}, \mathbb{R})$ and $\Psi \in D(H^{p/2})$. Then, for normalized Ψ,*

$$\langle :\Phi^2: \circ \gamma \rangle_\Psi (f^2) \leqslant B_{-p/2}(f) \| H^{p/2} \Psi \|^2 + C_0(f) \|\Psi\|^2 < \infty. \tag{91}$$

As an immediate consequence of theorem 5.3, we have the following.

Corollary 5.4. *Subject to the assumptions and notation of theorems 4.3 and 5.3, we have*

$$|\langle \mathfrak{Q}^\xi \rangle_\Psi (f)| \leqslant \big(\widetilde{\mathfrak{Q}}_A^\xi(f) + 2\xi C_0(\partial_t f) \big) \|\Psi\|^2 + 2\xi B_{-p/2}(\partial_t f) \| H^{p/2} \Psi \|^2. \tag{92}$$

It follows that there exists a constant $c'_{f,p}$ for which

$$(\rho^{\text{quant}} \circ \gamma)(f^2) \geqslant \mathfrak{Q}^\xi(f) \geqslant -c'_{f,p} (H + m \mathbb{1})^p \tag{93}$$

as an inequality of quadratic forms for Hadamard vector states Ψ.

In particular, we see that large negative time-averaged energy densities (for a given smearing function) can only be obtained at large positive overall energies.

As an application of these results, we may again consider the rescaled test function f_λ as in our earlier discussion of the AWEC. One can show that $\lambda C_0(\partial_t f_\lambda)$ and $\lambda B_{-p/2}(\partial_t f_\lambda)$ both tend to zero as $\lambda \to \infty$ (for the allowed values of p) which, together with our earlier observation that $\lambda \widetilde{\mathfrak{Q}}_A^\xi(f_\lambda) \to 0$, yields a proof of AWEC in the form (59) for all Hadamard vector states (which necessarily belong to the domain of $H^{p/2}$).

Now let us return to the four-dimensional case. Our result shows that a bound of the type (93) can be satisfied for any $p > 2$ (although we cannot exclude the possibility that it might also hold for some $p \leqslant 2$). The important point is that our QEI is a non-trivial restriction on the averaged energy density because the *upper* bound in (70) cannot be satisfied for any $q < 3$ (although it can be satisfied for any $q > 4$ by adapting our H-bound arguments).

6. Conclusion

Our main results may be summarized as follows. First, we have shown (at least for massless fields in four-dimensional Minkowski space) that the non-minimally coupled scalar field with $\xi > 0$ admits states with arbitrarily large negative energy density over arbitrarily large bounded spacetime regions. Thus, the non-minimally coupled field cannot obey QEIs of the type previously studied in the literature, in which there is a state-independent lower bound. Second, by combining the QEI derivation for the minimally coupled field [5] with techniques previously applied to the classical non-minimally coupled field [3], we have derived a new type of QEI with a state-dependent lower bound, for couplings in the range $(0, 1/4]$ in general globally hyperbolic spacetimes. Third, we have analysed the state-dependence of the bound, which involves the Wick square of the field, rather than the Wick powers of its derivatives that appear in the energy density. This involved the formulation of various H-bounds, which may be of independent interest. Roughly speaking, the results of this analysis tell us that the lower bound scales more softly with the overall energy scale than the energy density itself. Thus, we may conclude that negative energy effects with large magnitude, while possible over large regions, require more energy to achieve than positive energy densities of the same magnitude and that the 'energy budget' for these two effects will grow with a different power.

In the light of our results, it seems reasonable to expect that generic interacting quantum fields will not obey state-independent QEIs, as has also been argued on physical grounds for a particular model in [21]. The state-independent QEIs that have been found for other free fields and conformal fields in two dimensions (see [22]) should therefore be regarded as particularly simple cases. In general, then, the aim should be to establish non-trivial state-dependent bounds. Suitable generalizations of our H-bounds to higher order Wick polynomials may be useful in this context.

It also becomes important to understand whether one can draw useful physical conclusions from state-dependent QEIs. For example, the state-independent QEIs have been used to place constraints on exotic spacetimes [23–26] and are linked to questions of thermodynamic stability [27, 28]. A first indication that state-dependent bounds can be used for similar purposes may be found in our derivation of AWEC. In general, we expect that similar results will obtain for state-dependent bounds, once one restricts to states of a given energy scale. Likewise, we would expect the phenomenon of quantum interest [29, 30] to be governed by the energy scale in this context. Finally, our general QEI is of the so-called 'difference' type: it constrains the normal ordered energy density rather than the (Hadamard) renormalized version. This restriction has recently been removed for the minimally coupled field in [31] to obtain an 'absolute' QEI; one would expect that this can also be adapted to the non-minimally coupled field.

Appendix. Proof of theorem 4.1

Throughout the appendix $[\cdot]_c$ denotes the 'coincidence limit', i.e., the diagonal of smooth functions on $M \times M$, but for the sake of clarity, we will omit the arguments.

Proof of theorem 4.1. First, we look at the top line in (38). To make the calculations clearer, we start by calculating the expression without the symmetric product. We know by Synge's theorem that

$$\partial[F]_c = [(\partial \otimes \mathbb{1})F]_c + [(\mathbb{1} \otimes \partial)F]_c, \tag{A.1}$$

where F is a smooth function on $M \times M$, so

$$\begin{aligned}
&\partial[(\mathbb{1} \otimes \partial h^2)F - (\mathbb{1} \otimes h^2\partial)F]_c \\
&\quad = [(\partial \otimes \partial h^2)F + (\mathbb{1} \otimes \partial^2 h^2)F]_c - [(\partial \otimes h^2\partial)F + (\mathbb{1} \otimes \partial h^2\partial)F]_c \\
&\quad = [(\partial \otimes (\partial h^2))F + (\partial \otimes h^2\partial)F]_c \\
&\quad\quad + [(\mathbb{1} \otimes (\partial^2 h^2))F + 2(\mathbb{1} \otimes (\partial h^2)\partial)F + (\mathbb{1} \otimes h^2\partial^2)F]_c \\
&\quad\quad - [(\partial \otimes h^2\partial)F]_c - [(\mathbb{1} \otimes (\partial h^2)\partial)F + (\mathbb{1} \otimes h^2\partial^2)F]_c \\
&\quad = 2(\partial h^2)[(\mathbb{1} \otimes_{\mathfrak{s}} \partial)F]_c + (\partial^2 h^2)[(\mathbb{1} \otimes \mathbb{1})F]_c.
\end{aligned} \tag{A.2}$$

Where we want to emphasize that there is a symmetric product in the last line. Now it is quite obvious that the same calculation is valid if one exchanges the arguments of F. Thus, adding the term $2h^2[(\mathbb{1} \otimes_{\mathfrak{s}} \partial^2)F]_c$ on both sides, we find the following identity for the symmetrized expression:

$$\begin{aligned}
&2h^2[(\mathbb{1} \otimes_{\mathfrak{s}} \partial^2)F]_c + \partial[(\mathbb{1} \otimes_{\mathfrak{s}} \partial h^2)F - (\mathbb{1} \otimes_{\mathfrak{s}} h^2\partial)F]_c \\
&\quad = 2h^2[(\mathbb{1} \otimes_{\mathfrak{s}} \partial^2)F]_c + 2(\partial h^2)[(\mathbb{1} \otimes_{\mathfrak{s}} \partial)F]_c + (\partial^2 h^2)[(\mathbb{1} \otimes_{\mathfrak{s}} \mathbb{1})F]_c.
\end{aligned} \tag{A.3}$$

Here the top line is obviously identical with the expression in the top line of (38).

Now let us do the analogous calculations for the bottom line in (38). As before, we start for clarity without the symmetric product

$$\begin{aligned}
&2\partial[(h \otimes \partial h)F]_c - 2[(\partial h \otimes \partial h)F]_c + 2(\partial h)^2[(\mathbb{1} \otimes \mathbb{1})F]_c \\
&\quad = 2[(\partial h \otimes \partial h)F + (h \otimes \partial^2 h)F]_c - 2[(\partial h \otimes \partial h)F]_c + 2(\partial h)^2[(\mathbb{1} \otimes \mathbb{1})F]_c \\
&\quad = 2h[(\mathbb{1} \otimes (\partial^2 h))F + 2(\mathbb{1} \otimes (\partial h)\partial)F + (\mathbb{1} \otimes h\partial^2)F]_c + 2(\partial h)^2[(\mathbb{1} \otimes \mathbb{1})F]_c \\
&\quad = 2h^2[(\mathbb{1} \otimes \partial^2)F]_c + 2(2h\partial h)[(\mathbb{1} \otimes \partial)F]_c + (2h\partial^2 h + 2(\partial h)^2)[(\mathbb{1} \otimes \mathbb{1})F]_c \\
&\quad = 2h^2[(\mathbb{1} \otimes \partial^2)F]_c + 2(\partial h^2)[(\mathbb{1} \otimes \partial)F]_c + (\partial^2 h^2)[(\mathbb{1} \otimes \mathbb{1})F]_c.
\end{aligned} \tag{A.4}$$

Again, if we symmetrize this, we get

$$\begin{aligned}
&2\partial[(h \otimes_{\mathfrak{s}} \partial h)F]_c - 2[(\partial h \otimes_{\mathfrak{s}} \partial h)F]_c + 2(\partial h)^2[(\mathbb{1} \otimes_{\mathfrak{s}} \mathbb{1})F]_c \\
&\quad = 2h^2[(\mathbb{1} \otimes_{\mathfrak{s}} \partial^2)F]_c + 2(\partial h^2)[(\mathbb{1} \otimes_{\mathfrak{s}} \partial)F]_c + (\partial^2 h^2)[(\mathbb{1} \otimes_{\mathfrak{s}} \mathbb{1})F]_c.
\end{aligned} \tag{A.5}$$

Comparing the bottom row of (A.5) with the bottom row of (A.3) proves the theorem. □

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